ARL 89

ON SOME TRANSFORMATIONS IN THE ANALYSIS OF VARIANCE

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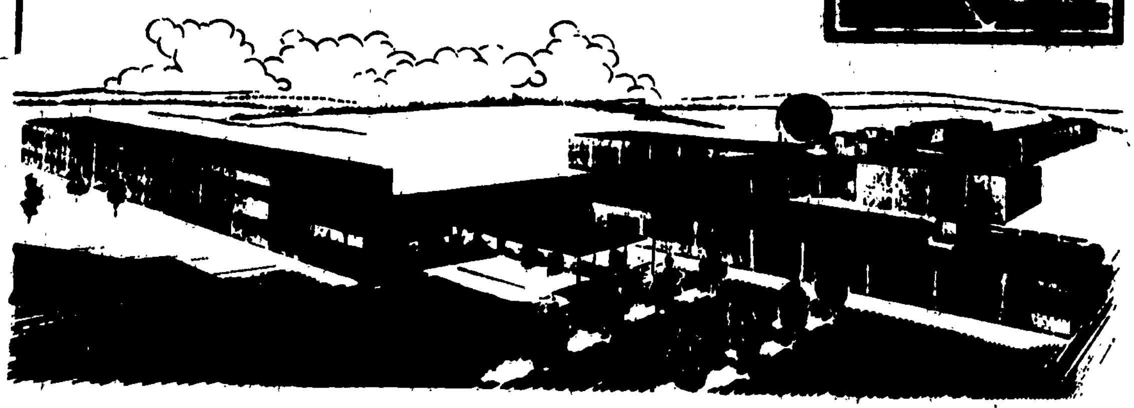
PITTSBURGH, PENNSYLVANIA

DECEMBER 1961

AERONAUTICAL RESEARCH LABORATORY
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE



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FOREWORD

This technical report was prepared by E. G. Olds, M. Branson, and R. E. Odeh* of the Department of Mathematics of Carnegie Institute of Technology under Contract No. AF 33(616)-3878, Task 70429, Transformations in the Analysis of Variance, on Project No. 7071, Mathematical Techniques of Aeromechanics. The work on this task was administered under the direction of the Aeronautical Research Laboratory, Office of Aerospace Research, Wright-Patterson Air Force Base, Ohio, with Dr. H. Leon Harter as task scientist. Several Technical Reports and Notes (see references [26], [28], [29], [30], and [34]) have previously been issued under this contract and its predecessor, Contract AF 33(616)-2418.

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ABSTRACT

Details are given of further results which have been obtained for the Log-variance test applied to the Analysis of Variance of Variances; an alternative method, the Log-range test, is proposed. Transformations in the Analysis of Variance are discussed, and a test is proposed for deciding whether or not to transform the data. Finally, investigation into the problems when the sample variates are not independent is mentioned. The topics included in this discussion are (i) transformation of the variates, and (ii) effect upon the distribution of sample range.

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INTRODUCTION

This report is concerned with work on a project having the general title "Research on Transformations in the Analysis of Variance". The project was initiated with the purpose of investigating the following topics:

1. The theory of various transformations in the Analysis of Variance, including: square-root, logarithmic and reciprocal. This investigation shall include a study of objective tests which the experimenter might employ as criteria for determining the type of transformation to use, and of the possibility of devising better tests. Consideration shall be given to the effects on the final interpretation of the data of (a) applying an unnecessary transformation and (b) failure to apply a necessary transformation.

2. The relative importance of homogeneity of variance and of additivity of effects in the Analysis of Variance. This investigation shall be directed toward answering the questions as to whether, prior to performing an Analysis of Variance, one should transform the data so as to (i) equalize the variances, (ii) minimize the ratio of the mean square for Tukey's one degree of freedom for non-additivity to the residual mean square, or (iii) endeavor to make non-significant the departures from both homogeneity and additivity.

3. The procedure for the Analysis of Variance applied to variances with particular attention given to the transformation required, to the optimum division of the observations into subgroups, and to the power of the resulting tests. In particular, a Monte Carlo study shall be made of the Analysis of Variance of log s^2 for samples of a

particular size, subdivided in various ways.

4. The best procedure for the Analysis of Variance of attributes data (binomially distributed). Consideration shall be given to the relative merits of logit, probit, and anglit transformations. A comparison of factorial chi-square tests and conventional F-tests shall be made. The effects of the transformation on the data source will be emphasized in this investigation, rather than the effects on the data themselves.

5. Procedures designed to produce normality. In particular, consideration shall be given to (i) the transformation to standard normal scores and (ii) procedures which assume that $y = (x + c)^p$ is normally distributed and estimate c and p by (a) the method of moments and by (b) the method of maximum likelihood.

Several Technical Reports and Notes have been issued previously. The first report [29] considered general problems of transformations in the Analysis of Variance, whereas [30] and [34] concentrated mainly upon the logarithmic and square-root transformations (Topic 1). The Analysis of Variance of Attributes Data (Topic 4) was discussed in [28], and one note [26] has so far been issued on the Analysis of Variance of Variances (Topic 3).

The present report describes research that has been carried out on this project since the above mentioned Technical Reports and Notes were issued. The work discussed here consists mainly of further research into the problem of Analysis of Variance of Variances, together with a section given to the problem of deciding whether or not to transform the data before carrying out an

Analysis of Variance. It is hoped that separate, and more detailed, Technical Documentary Reports based on these investigations will be issued shortly.

1. THE ANALYSIS OF VARIANCE OF VARIANCES.

1.1. <u>The Problem of Heterogeneity of Variance.</u> The standard one-way Analysis of Variance model for means, with equal sample sizes, has the form

$$y_{kj} = \mu_k + \varepsilon_{kj} \qquad \begin{array}{l}(j = 1,2,\ldots,J)\\(k = 1,2,\ldots,K)\end{array} \qquad \ldots(1.1)$$

where y_{kj} is the j-th observation from the k-th population,

μ_k is the true mean of the k-th population, and

ε_{kj} is a random variable with mean zero and variance σ_k^2.

J is the number of observations taken from each of K populations.

The decision to accept or reject the null hypothesis, $H_0 : \mu_k = \mu$ (μ unspecified) for all k, at a significance level α, is made by comparing the magnitude of the F-ratio (that is, the ratio of the Between Groups Mean Square to the Within Groups Mean Square) with a preassigned significance point F_α. Now the calculation of the distribution of the F-ratio, and hence of the significance points F_α, depends upon the assumption that ε_{kj}, in the above model, are normal independent deviates with zero mean and common variance σ^2.

It is therefore necessary to be able to test this assumption before placing reliance upon results that may be obtained from the Analysis of Variance.

A previous report entitled "Notes on the Analysis of Variance of Logarithms of Variances" [26] described a procedure for testing whether the variances of ε_{kj} are equal for all of the K

populations. A summary of this report is included below in order that the more recent work described may be readily followed.

1.2. The Analysis of Variance of Logarithms of Variances. It has been suggested (Box: Biometrika, 40, 1953, pp. 318-335 [5]) that the Bartlett test for the equality of variances is very sensitive to departure from normality as well as to the heterogeneity of variance. On the other hand, the F-test obtained in the Analysis of Variance for means is relatively "robust" with respect to departures from normality per se, at least for the case with equal sample sizes, whilst it is affected seriously by variance heterogeneity. Thus it was desired to obtain a test that would be far less dependent upon the normality assumption than is the Bartlett test.

The procedure proposed is to divide the observations within each group into subgroups, apply a logarithmic transformation to the subgroup variances, and then to perform an Analysis of Variance on the logarithms. An example of the method is worked, and a justification of the procedure is detailed.

The method is as follows:

1) Divide the observations y_{kj} within each of the K groups into M subgroups of size A ($MA = J$, $M > 1$, $A > 1$), the division being carried out by a randomizing procedure.

2) Denote by x_{kma} the a-th observation in the m-th subgroup of the k-th group. Then calculate the sum of the squares of deviations from the mean for each subgroup, i.e., calculate

$$\sum_{a=1}^{A} (x_{kma} - \bar{x}_{km.})^2$$

where $\bar{x}_{km.} = \sum_{a=1}^{A} x_{kma}/A$ is

the mean of the data in the m-th subgroup of the k-th group.

3) Calculate the logarithm of the above sum of squares, call this variate z_{km}. Thus

$$z_{km} = \log_{10} [\sum_{a=1}^{A} (x_{kma} - \bar{x}_{km.})^2] . \qquad \ldots(1.2)$$

4) Carry out the standard Analysis of Variance technique on the z_{km}. Then if we denote the between Groups Sum of Squares by S_B^2, and the Within Groups Sum of Squares by S_W^2, the F-ratio, F_L, is obtained, where

$$F_L = \frac{S_B^2/(K-1)}{S_W^2/K(M-1)} \qquad \ldots(1.3)$$

Clearly, since the z_{km} are not distributed normally, the distribution of F_L will not be exactly that of a "normal theory" F. However, several approximations were considered, and approximate percentage points obtained.

Since the technical note [26], summarized above, was issued further study has been given to the problem. This will now be described below.

1.3. Power of the Analysis of Variance of Logarithms of Variances Test. The power of a statistical test is defined as the probability of rejecting the null hypothesis when it is false; that is, the probability of reaching a correct decision when the null hypothesis is not true. Thus if we denote the null hypothesis by H_0, and any given alternative by H', then the power with respect to H' is given by

$$\text{Power} = P\left\{\text{rejecting } H_0 \middle| H' \text{ is true}\right\}.$$

It was not possible to obtain the exact distribution of F_L either under the null hypothesis or under any alternative. Thus, in order to investigate the power of the test, it was necessary to approximate the distribution. Now the quantity $P\left\{F_L > F_\alpha \middle| H_0\right\}$ represents the probability of Type I error, when this probability is nominally α. Clearly the probability will not in general be exactly α, since F_L is not distributed exactly as F. Similarly, $P\left\{F_L > F_\alpha \middle| H'\right\}$ will give an approximation to the power of the test.

The investigation of the power of the test was confined to consideration of the two following forms of alternative:

a) The variance of one of the K populations is equal to $\phi'\sigma^2$. The variance of each of the remaining K-1 populations is equal to σ^2.

b$_1$) For K even, the variance of K/2 of the populations is equal to $\phi'\sigma^2$. The variance of each of the remaining K/2 populations is equal to σ^2/ϕ'.

b$_2$) For K odd, the variance of each of (K-1)/2 of the

populations is $\phi'\sigma^2$. The variance of each of (K-1)/2 populations is equal to σ^2/ϕ'. The variance of the remaining population is σ^2.

Approximations to the power curves have been calculated for K = 2, 5, 10, and 15 with sample sizes of 12 and 24. The samples have been divided into subgroups of all possible equal sizes in order to ascertain which form of subdivision results in the greatest power being obtained. (See Tables 1 - 12) The method of approximation used was that suggested by David and Johnson, [11] and [10]. The frequency curves used for this approximation were:

K = 2, K = 5: Pearson Type IV curve (except for the underlined values which were calculated from a Pearson Type VI curve)

K = 10, K = 15 : Edgeworth series (using the first four terms). For K = 2, K = 5 some values are omitted from the Tables, the reason being that for these cases computing time would have been too great. The values obtained were usually accurate to two decimal places at least, and, since this degree of accuracy is sufficient for making the necessary power comparisons in this investigation, all the values have been listed to two decimal places.

TABLES 1 - 12*

Power (approximate) of the Analysis of Variance of Logarithms of Variances Test when the nominal significance level is 5 per cent and when there are K groups each containing M subgroups of size A, for various values of ϕ'.

* The notation used to head the Tables will be:

$$P\left\{F_L > F_{.05} \middle| \phi'; K, MA\right\}$$

TABLE 1

$$P\left\{F_L > F_{.05} \middle| \phi' ;\ K = 2\ ,\ MA = 12\right\}$$

Alternatives of Type (a)

ϕ' \ (M,A)	(6,2)	(4,3)	(3,4)	(2,6)
1.0	.05	.05	.04	
1.5	.08	.09	.09	
2.0	.11	.14		.12
2.5	.14	.19	.20	.17
3.0	.17	.24	.25	.20
3.5	.20	.28	.29	.24
4.0	.23	.32	.33	.27
4.5	.25	.36	.37	.29
5.0	.28	.39	.40	.32
5.5	.30	.42	.44	.34
6.0	.32	.45	.46	.36
7.0	.36	.50	.52	.39
8.0	.39	.55	.56	.42
9.0	.42	.59	.60	.45
10.0	.45	.62	.63	.47
12.0	.50	.67	.69	.51
14.0	.54	.72	.73	.55
16.0	.57	.75	.76	.58
18.0	.60	.78	.79	.60
20.0	.63	.80	.81	.62

TABLE 2

$$P\left\{F_L > F_{.05} \middle| \phi' ;\ K = 2\ ,\ MA = 24\right\}$$

Alternatives of Type (a)

ϕ' \ (M,A)	(12,2)	(8,3)	(6,4)	(4,6)	(3,8)	(2,12)
1.0	.06	.06	.06	.06	.07	.06
1.5	.09	.11	.12	.13	.14	.12
2.0	.14	.21	.24	.25	.25	.20
2.5	.20	.31	.36	.38	.35	.27
3.0	.26	.41	.48	.50	.46	.33
3.5	.32	.50	.57	.59	.55	.38
4.0	.37	.57	.65	.67	.62	.43
4.5	.41	.63	.71	.73	.69	.47
5.0	.46	.68	.75	.78	.74	.51
5.5	.49	.72	.79	.82	.78	.54
6.0	.53	.75	.82	.85	.81	.57
7.0	.59	.81	.87	.89	.86	.62
8.0	.63	.85	.91	.92	.90	.66
9.0	.67	.88	.93	.94	.92	.70
10.0	.71	.90	.95	.96	.94	.73
12.0	.76	.93	.97	.97	.96	.78
14.0	.80	.95	.98	.98	.97	.82
16.0	.83	.97	.98	.99	.98	.85
18.0	.86	.97	.99	.99	.99	.87
20.0	.87	.98	.99	.99	.99	.89

TABLE 3

$$P\left\{F_L > F_{.05} \middle| \phi' ; K = 5 , MA = 12\right\}$$

Alternatives of Type (a)

ϕ' \ (M,A)	(6,2)	(4,3)	(3,4)	(2,6)
1.0	.05			.06
1.5	.06		.07	.07
2.0	.07	.10	.11	.10
2.5	.09	.13	.15	.14
3.0	.11	.17	.19	
3.5	[illegible]	.21	.24	.22
4.0	[illegible]	.25	.29	.25
4.5		.30	.33	.29
5.0	.20	.34	.37	.32
5.5		.37	.41	.35
6.0		.41	.45	.38
7.0		.47	.52	.44
8.0		.53	.58	.49
9.0		.58	.63	.53
10.0	.39	.62	.67	.57
12.0		.68	.73	.63
14.0		.73	.78	.69
16.0		.77	.82	.73
18.0		.80	.85	.76
20.0	.60	.83	.87	.79

TABLE 4

$$P\left\{F_L > F_{.05} \middle| \phi'; K = 5, MA = 24\right\}$$

Alternatives of Type (a)

ϕ' \ (M,A)	(12,2)	(8,3)	(6,4)	(4,6)	(3,8)	(2,12)
1.0	.05	.05	.05	.05	.05	.05
1.5	.06	.08	.09	.10	.10	.10
2.0	.10	.15	.19	.21	.20	.18
2.5	.14	.25	.31	.34	.33	.26
3.0	.20	.36	.44	.48	.46	.35
3.5	.25	.45	.55	.60	.57	.43
4.0	.30	.54	.64	.69	.67	.51
4.5		.61	.71	.76	.74	.58
5.0	.40	.67	.77	.81	.80	.64
5.5		.72		.86	.84	.69
6.0		.76		.89	.87	.73
7.0		.83		.93	.92	.80
8.0				.95	.95	.85
9.0				.97	.96	.89
10.0	.63			.98	.98	.91
12.0				.99	.99	.94
14.0				.99	.99	.96
16.0				1.00	1.00	.98
18.0				1.00	1.00	.98
20.0			.96	1.00	1.00	.99

TABLE 5

$$P\left\{F_L > F_{.05} \middle| \phi'; K = 10, MA = 12\right\}$$

Alternatives of Type (a)

ϕ' \ (M,A)	(6,2)	(4,3)	(3,4)	(2,6)
1.0	.05	.04	.04	.05
1.5	.05	.05	.05	.06
2.0	.06	.07	.08	.08
2.5	.07	.10	.11	.11
3.0	.09	.13	.15	.15
3.5	.10	.16	.18	.18
4.0	.12	.19	.22	.22
4.5	.13	.23	.27	.25
5.0	.15	.26	.31	.29
5.5	.17	.29	.34	.32
6.0	.18	.33	.38	.36
7.0	.22	.39	.45	.42
8.0	.25	.44	.51	.48
9.0	.28	.49	.56	.53
10.0	.31	.54	.61	.57
12.0	.36	.61	.69	.64
14.0	.41	.67	.75	.70
16.0	.45	.72	.79	.75
18.0	.49	.76	.82	.79
20.0	.52	.79	.85	.82

TABLE 6

$$P\left\{F_L > F_{.05} \,\middle|\, \phi'; \; K = 10 \, , \; MA = 12\right\}$$

Alternatives of Type (b)

ϕ' \ (M,A)	(6,2)	(4,3)	(3,4)	(2,6)
1.0	.05	.04	.04	.05
1.5	.11	.18	.21	.21
2.0	.30	.53	.61	.57
2.5	.52	.81	.87	.83
3.0	.71	.93	.96	.94
3.5	.82	.98	.99	.98
4.0	.90	.99	1.00	.99
4.5	.94	1.00	1.00	1.00
5.0	.96	1.00	1.00	1.00

TABLE 7

$$P\left\{F_L > F_{.05} \middle| \phi' ; K = 10 , MA = 24\right\}$$

Alternatives of Type (a)

ϕ' \ (M,A)	(12,2)	(8,3)	(6,4)	(4,6)	(3,8)	(2,12)
1.0	.05	.05	.05	.05	.05	.05
1.5	.06	.07	.08	.08	.08	.08
2.0	.08	.12	.15	.16	.16	.14
2.5	.11	.19	.25	.27	.27	.23
3.0	.15	.28	.36	.40	.40	.32
3.5	.19	.37	.46	.52	.51	.42
4.0	.23	.45	.56	.62	.61	.51
4.5	.27	.53	.64	.70	.70	.58
5.0	.31	.59	.71	.77	.76	.65
5.5	.35	.65	.76	.82	.81	.71
6.0	.39	.70	.81	.86	.85	.76
7.0	.46	.78	.87	.91	.91	.83
8.0	.52	.83	.91	.95	.94	.88
9.0	.57	.87	.94	.97	.96	.91
10.0	.62	.90	.96	.98	.98	.94
12.0	.70	.94	.98	.99	.99	.96
14.0	.75	.96	.99	.99	.99	.98
16.0	.80	.97	.99	1.00	1.00	.99
18.0	.83	.98	1.00	1.00	1.00	.99
20.0	.86	.99	1.00	1.00	1.00	1.00

TABLE 8

$$P\left\{F_L > F_{.05} \middle| \phi'; K = 10, MA = 24\right\}$$

Alternatives of Type (b)

ϕ' \ (M,A)	(12,2)	(8,3)	(6,4)	(4,6)	(3,8)	(2,12)
1.00	.05	.05	.05	.05	.05	.05
1.05	.05	.05	.06	.05	.05	.06
1.10	.06	.06	.07	.07	.07	.07
1.20	.07	.10	.12	.13	.13	.12
1.30	.10	.17	.22	.25	.25	.21
1.40	.15	.28	.36	.41	.41	.34
1.50	.21	.42	.53	.59	.58	.48
1.60	.28	.56	.68	.74	.73	.62
1.70	.37	.68	.80	.85	.85	.75
1.80	.45	.78	.88	.92	.92	.84
1.90	.54	.86	.94	.96	.96	.90
2.00	.62	.91	.97	.98	.98	.94
2.20	.75	.97	.99	1.00	1.00	.98
2.50		.99	1.00	1.00	1.00	1.00
3.00		1.00	1.00	1.00	1.00	1.00

TABLE 9

$$P\left\{F_L > F_{.05} \middle| \phi'; K = 15, MA = 12\right\}$$

Alternatives of Type (a)

ϕ' \ (M,A)	(6,2)	(4,3)	(3,4)	(2,6)
1.0	.05	.05	.05	.05
1.5	.05	.06	.06	.05
2.0	.06	.07	.07	.07
2.5	.07	.09	.10	.10
3.0	.08	.11	.13	.12
3.5	.09	.14	.16	.15
4.0	.10	.17	.19	.18
4.5	.11	.19	.23	.21
5.0	.12	.22	.26	.24
5.5	.14	.25	.30	.28
6.0	.15	.28	.33	.31
7.0	.17	.33	.39	.36
8.0	.20	.39	.45	.42
9.0	.22	.43	.51	.47
10.0	.25	.48	.55	.51
12.0	.29	.55	.63	.59
14.0	.33	.61	.70	.66
16.0	.37	.67	.75	.71
18.0	.40	.71	.79	.75
20.0	.44	.74	.82	.78

TABLE 10

$$P\left\{F_L > F_{.05} \middle| \phi'; K = 15 , MA = 12\right\}$$

Alternatives of Type (b)

ϕ' \ (M,A)	(6,2)	(4,3)	(3,4)	(2,6)
1.00	.05	.05	.05	.05
1.25	.07	.09	.09	.09
1.50		.21	.25	.23
1.75		.41	.48	.45
2.00		.62	.71	.67
2.25		.79	.86	.83
2.50		.89	.94	.92
2.75		.95	.97	.96
3.00		.97	.99	.98
3.25		.99	1.00	.99
3.50		.99	1.00	1.00

TABLE 11

$$P\left\{F_L > F_{.05} \middle| \phi' ; K = 15 , MA = 24\right\}$$

Alternatives of Type (a)

ϕ' \ (M,A)	(12,2)	(8,3)	(6,4)	(4,6)	(3,8)	(2,12)
1.0	.06	.05	.05	.05	.05	.05
1.5	.06	.07	.07	.08	.08	.07
2.0	.08	.10	.12	.14	.14	.12
2.5	.11	.16	.20	.24	.24	.19
3.0	.14	.23	.29	.35	.35	.28
3.5	.17	.30	.39	.46	.46	.37
4.0	.21	.38	.48	.56	.56	.45
4.5	.24	.45	.56	.65	.64	.53
5.0	.28	.51	.64	.72	.72	.60
5.5	.31	.57	.70	.78	.77	.66
6.0	.35	.63	.75	.82	.82	.72
7.0	.41	.71	.83	.89	.89	.80
8.0	.47	.78	.88	.93	.93	.86
9.0	.52	.83	.91	.95	.95	.90
10.0	.57	.87	.94	.97	.97	.93
12.0	.65	.92	.97	.99	.99	.96
14.0	.71	.95	.98	.99	.99	.98
16.0	.76	.96	.99	1.00	1.00	.99
18.0	.80	.97	.99	1.00	1.00	.99
20.0	.83	.98	1.00	1.00	1.00	1.00

TABLE 12

$$P\left\{F_L > F_{.05} \middle| \phi' ; K = 15 \text{ , } MA = 24\right\}$$

Alternatives of Type (b)

ϕ' \ (M,A)	(12,2)	(8,3)	(6,4)	(4,6)	(3,8)	(2,12)
1.00	.06	.05	.05	.05	.05	.05
1.25		.14	.18	.21	.21	.17
1.50		.48	.60	.69	.69	.57
1.75		.82	.91	.95	.95	.89
2.00		.96	.99	1.00	1.00	.98
2.25		.99	1.00	1.00	1.00	1.00
2.50	.95	1.00	1.00	1.00	1.00	1.00

1.4. <u>The Choice of Subgroup Size</u>. One of the purposes of this investigation was to determine the optimum choice of subgroup size. It is desirable to make the choice in such a way that the resulting test is "best" in the following sense:

Let $T_c(M_c, A_c)$ denote the test procedure generated by choosing M_c subgroups of size A_c,

$$\text{where } M_c A_c = J$$
$$c = 1, 2, \ldots, C$$

and where C is the total number of possible subdivisions. Denote by α_c the probability of Type I error for the test procedure T_c, when the nominal probability of such error is α.

Let $P_c(\phi')$ denote the power of T_c against a particular alternative determined by ϕ'.

Then we shall say that $T_{c'}(M_{c'}, A_{c'})$ is best if

$$\alpha_{c'} \leq \alpha_c$$
$$\text{and} \quad P_{c'}(\phi') \geq P_c(\phi')$$

for all $c = 1, 2, \ldots, c' - 1,\ c' + 1, \ldots, C$, and for all ϕ'.

Since the results given in 1.3 are numerical in nature, and only two different alternatives have been considered, it will not be possible to say that one particular mode of subdivision is the best in the above sense. We can, however, make some general remarks concerning these results.

It is interesting to note that corresponding to an increase in subgroup size, or to an increase in the number of subgroups, there is an increase in power. The former effect is due to the fact that the variance of the variate z decreases as the

subgroup size increases. The latter effect is due to the fact that the number of degrees of freedom in the denominator of the F-ratio increases as the number of subgroups is increased. These effects may be seen to hold true for the cases considered numerically in 1.3, and will be shown to be true asymptotically in section 1.7 below. However, in the problem under consideration, the total sample size is constant, thus an increase in subgroup size must be accompanied by a decrease in the number of subgroups, and vice versa. It is therefore necessary to balance these two effects in the most advantageous manner.

On the basis of the numerical results, it would appear that for the cases considered in Tables 1 - 12 we should make the following subdivisions:

(i) For samples of size 12: 3 subgroups of size 4,

(ii) For samples of size 24: 4 subgroups of size 6, or 3 subgroups of size 8.

However it might be dangerous to formulate a general rule from the few particular cases considered here.

1.5. Comparison of Power with Standard F-test. It has been suggested (see Scheffé [36], p. 86) that the power of the Analysis of Variance of Logarithms of Variances test, which we shall hereafter refer to as the L. V. test, can be computed by noting that F_L has approximately a non-central F-distribution. Therefore, it is of interest to compare the power of the L. V. test computed in 1.3 with the power computed by assuming that z_{km} is exactly normally distributed and using Tang's Tables [38] or, equivalently, the

Pearson and Hartley charts [31] for the power of the F-test.

It is convenient to rewrite (1.2) (the logarithms to the base $_{10}$ may be replaced by natural logarithms, since this merely alters the value of z_{km} by a constant multiplier) as

$$z_{km} = \ln \frac{ns^2}{\sigma_k^2} + \ln \sigma_k^2$$

$$\text{where } s^2 = \frac{1}{n} \sum_{a=1}^{A} (x_{kma} - \bar{x}_{km.})^2$$

and where $n = A - 1$.

Then

$$z_{km} = E[\ln \frac{ns^2}{\sigma_k^2}] + \ln \sigma_k^2 + [\ln \frac{ns^2}{\sigma_k^2} - E(\ln \frac{ns^2}{\sigma_k^2})]$$

$$= \mu' + \beta_k + \epsilon'_{km} \qquad \ldots(1.4)$$

where

$$\mu' = E[\ln \frac{ns^2}{\sigma_k^2}] + \frac{1}{K} \sum_{k=1}^{K} \ln \sigma_k^2$$

$$\beta_k = \ln \sigma_k^2 - \frac{1}{K} \sum_{k=1}^{K} \ln \sigma_k^2$$

$$\text{and } \epsilon'_{km} = [\ln \frac{ns^2}{\sigma_k^2} - E(\ln \frac{ns^2}{\sigma_k^2})] \quad . \qquad \ldots(1.5)$$

It follows that

$$E(\epsilon'_{km}) = 0 \qquad \ldots(1.6)$$

and that

$$\sum_{k=1}^{K} \beta_k = 0 \; . \qquad \ldots(1.7)$$

The hypothesis H_0 of variance homogeneity and the alternative H_1 may now be written, in terms of the above notation, as

$$H_0 : \sum_{k=1}^{K} (\beta_k)^2 = 0 \qquad \ldots(1.8)$$

$$H_1 : \sum_{k=1}^{K} (\beta_k)^2 \neq 0 \; . \qquad \ldots(1.9)$$

It is easily shown that the expectations of the mean squares which occur in the F_L-ratio, (1.3), are

$$E[S_B^2/(K-1)] = \sigma_L^2 + \frac{M}{(K-1)} \sum_{k=1}^{K} (\beta_k)^2 \qquad \ldots(1.10)$$

$$E[S_W^2/K(M-1)] = \sigma_L^2 \qquad \ldots(1.11)$$

where σ_L^2 = Var [ln s^2] for subgroups of size A.

It is well known that when ε'_{km} has a normal distribution, the ratio of mean-squares is distributed, under the alternative hypothesis, as a non-central F,

$$F'_{K-1,\; K(M-1)}(\delta)$$

where the square of the non-centrality parameter δ is given by

$$\delta^2 = \frac{M}{\sigma_L^2} \sum_{k=1}^{K} (\beta_k)^2 \qquad \ldots(1.12)$$

Tables of $P\left\{F'_{\nu_1,\nu_2} > F_\alpha\right\}$ are given by Tang [38] and charts are given by Pearson and Hartley [31], as mentioned above. In Tang's notation, we introduce a quantity ϕ, as a measure of non-centrality, where

$$\phi = \frac{\delta}{(\nu_1 + 1)^{1/2}} = \frac{\delta}{K^{1/2}} \qquad \ldots(1.13)$$

For alternatives of "single slippage" [Type (a)] it is easily shown that

$$\sum_{k=1}^{K} (\beta_k)^2 = \frac{K-1}{K} \ln^2 \phi' \qquad \ldots(1.14)$$

from which it is seen that

$$\delta = \frac{1}{\sigma_L} \sqrt{\frac{M(K-1)}{K}} \ln \phi' \qquad \ldots(1.15)$$

whence $$\phi = \frac{\sqrt{M(K-1)}}{K\,\sigma_L} \ln \phi' \qquad \ldots(1.16)$$

For alternatives of "equal multiple slippage" [Type (b)] it can similarly be shown that

(i) For K even, $$\delta = \frac{\sqrt{MK} \ln \phi'}{\sigma_L}$$

and $$\phi = \frac{\sqrt{M} \ln \phi'}{\sigma_L} \qquad \ldots(1.17)$$

(ii) For K odd,

$$\delta = \frac{\sqrt{M(K-1)}}{\sigma_L} \ln \phi'$$

$$\text{and} \qquad \phi = \frac{1}{\sigma_L}\sqrt{\frac{M(K-1)}{K}} \ln \phi' \qquad \ldots(1.18)$$

Table 13 compares the power of the L.V. test computed by the method of David and Johnson with the power computed from Tang's tables. For K = 2, the power is given for 3 subgroups of size 4. For K = 5, the comparison is given for 4 subgroups of size 6.

TABLE 13

Comparison of two methods of computing the power of the L.V. test

Power when the nominal significance level is .05.

ϕ	K=2 David/Johnson	K=2 Tang	K=5 David/Johnson	K=5 Tang
1.0	.24	.20	.31	.30
1.5	.42	.37	.63	.62
2.0	.60	.57	.88	.88
2.5	.75	.75	.98	.98

It should be noted that the results obtained from Tang's tables were found under the assumption that the ε'_{km} are normally distributed. Thus the differences observed between the two methods of computation in Table 13 illustrate the effect of the ε'_{km} being, in fact, not normally distributed. It will be noticed that the effect of non-normality is greater for K = 2 than it is for K = 5. In fact, it will be seen in Section 1.7 that as either the number of subgroups, or the subgroup size, is increased the distribution of the test statistic F_L becomes nearer to that which would be obtained

under normal theory.

1.6. Comparison of the Power of the L.V. Test with that of Other Tests of Homogeneity. One of the standard tests for equality of variances is Bartlett's test. Box [5] has shown that this test is very sensitive to the assumption that the sampled population is normal. Indeed it has been suggested that the method serves equally well as a test for normality as for testing equality of variances.

Box and Andersen [6] have suggested a modified form of Bartlett's statistic which appears to make the test more robust. Their test statistic is of the form

$$M' = \frac{M}{1 + \frac{k_4}{2k_2^2}} \qquad \ldots(1.19)$$

where M is Bartlett's statistic, and k_2 , k_4 are Fisher's k-statistics.

They also performed a sampling experiment in order to compare the power of the test using M with that using M'. Samples of size 20 were taken from each of 10 normal populations. The following alternative hypothesis was considered:

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2 = 1$$

$$\sigma_5^2 = \sigma_6^2 = \sigma_7^2 = 1.7$$

$$\sigma_8^2 = \sigma_9^2 = \sigma_{10}^2 = 2.6 \qquad \ldots(1.20)$$

An alternative procedure has been suggested by Levene [23]. His method is to apply one of the following transformations:

$$T_1 = |y_{kj} - \bar{y}_{k.}|$$

$$T_2 = (T_1)^2$$

$$T_3 = \log_{10} T_1$$

$$T_4 = \sqrt{T_1} \qquad \ldots(1.21)$$

(The above transformations are in the notation of our discussion. Levene used a notation: $T_1 = z_{ij}$; $T_2 = s_{ij}$; $T_3 = L_{ij}$; $T_4 = t_{ij}$.) Then, using T_i ($i = 1,2,3$, or 4), carry out a one-way Analysis of Variance.

Levene did an extensive sampling experiment to estimate the power of the tests using the above transformations. In sampling experiments with $K = 2$, and samples of size 20, T_3 had extremely poor power and, therefore, was not included in any further experiments. For $K = 10$, he obtained one thousand values of F under the null hypothesis and under the hypothesis given by (1.20) above.

In the present investigation, the power has been considered both for the L.V. test and the L.R. test* under the same alternative. In this particular case the samples of size 20 were each divided into 4 subgroups of size 5, since this subdivision appeared to give a better power than any other.

* L.R. test is an abbreviation for Log-range test. This test is suggested as an alternative to the L.V. test, and is discussed in section 1.8.

Olds and Kennard [27] have studied the control chart for ranges as a test for homogeneity. They conducted a relatively small sampling experiment (25 control charts) for the hypothesis given by (1.20) with the purpose of comparing their results with those of Box and Andersen.

Table 14 gives a comparison of the power of these various tests under the hypothesis given by (1.20).

TABLE 14

Power comparisons of some tests for homogeneity when the nominal significance level is .05

	Power
Bartlett	.815
Modified Bartlett	.810
T_1	.680
T_2	.656
T_4	.577
L.V.	.48
C.C. for Range	.44

As seen from the table the modified Bartlett test does have very good power compared to that of the Bartlett test. There is a considerable loss in power for the L.V. test and the control chart for ranges. It is thought that the L.R. test has less power than the L.V. test, but this matter seems to require further investigation. For the L.V. test the loss of power may be compensated by the fact that the test would be expected to be "robust".

The tests based on T_i have good power and have the

advantage that they are easier to apply than the Bartlett or modified Bartlett test. From a theoretical point of view they are difficult to investigate, since they have the property that, under any given alternative hypothesis, the transformed variate will have not only different means, but also different variances, for each population. Also, the transformed variates will no longer be independent.

It is seen that against the alternative considered, the L.V. test and the L.R. test have considerably less power than either of the Bartlett tests, or any of the T_j tests ($j = 1, 2, 4$). However, there is some evidence to suggest that all of these last mentioned tests are less robust than the L.V. test with respect to departures from underlying normality. It is clear that there is a need for further investigation on this subject.

1.7. Some Asymptotic Results for the L.V. Test. The asymptotic distribution of the test statistic F_L is now considered as K, M, or A tends to infinity. It is assumed that the density function of the z_{km} satisfies certain regularity conditions and has at least the first four moments finite. The following definitions are needed here (See Scheffé [36],p. 412):

Definition 1. If $x_1,\ldots, x_\nu$ are normal independent variables with x_i having mean ξ_i and unit variance, then the random variable

$$\chi'^2_\nu(\delta) = \sum_{i=1}^{\nu} x_i^2$$

is called a non-central chi-square variable with ν degrees of

freedom. The quantity δ is known as the non-centrality parameter, where

$$\delta^2 = \sum_{i=1}^{\nu} \xi_i^2 .$$

It should be noted that when δ is zero the distribution reduces to that of a central chi-square, χ_ν^2.

<u>Definition</u> 2. If $\chi_{\nu_1}'^2(\delta)$ and $\chi_{\nu_2}^2$ are independent random variables with distributions as defined in Definition 1, then the distribution of the ratio of these quantities divided by their degrees of freedom,

$$F'_{\nu_1, \nu_2}(\delta) = \frac{\chi_{\nu_1}'^2(\delta)/\nu_1}{\chi_{\nu_2}^2/\nu_2}$$

is called a non-central F-distribution with ν_1 and ν_2 degrees of freedom, and non-centrality parameter δ.

We shall now quote some lemmas which will be useful in our discussion of the limiting distributions of the test statistic, F_L. "Distribution function" is used here in the cumulative sense.

<u>Lemma</u> 1: (Cramér [8], sec. 20.6) Let $u_1, u_2, \ldots$ be a sequence of random variables, with distribution functions $F_1, F_2, \ldots$ Suppose that $F_n(x)$ tends to a distribution function $F(x)$ as $n \longrightarrow \infty$.

Let $v_1, v_2, \ldots$ be another sequence of random variables, and suppose that v_n converges in probability to a constant c.

Let $w_n = u_n v_n$. Then the distribution function of w_n tends to $F(x/c)$.

__Lemma__ 2: (Cramér [8], sec. 27.3.) Let $x_1, x_2, \ldots, x_N$ be a sequence of independent observations from a population with distribution function $F(x)$.

Let $a_\nu^{(N)} = \frac{1}{N} \sum_{i=1}^{N} x_i^\nu$ denote the ν-th sample moment, and let $\alpha_\nu = E[a_\nu^{(N)}]$. Let g be any rational function, or power of a rational function.

If $\alpha_1, \ldots, \alpha_k$ are finite and $g(\alpha_1, \ldots, \alpha_k)$ is defined, then

$$g[a_1^{(N)}, \ldots, a_k^{(N)}] \longrightarrow g(\alpha_1, \ldots, \alpha_k) \text{ in probability,}$$

as $N \longrightarrow \infty$.

__Lemma__ 3: (Due to Mann and Wald, [24], Theorem 5. The following statement of the theorem is given by Rao [34], Lemma 3.)

Suppose $F_N(u_1, \ldots, u_k)$ is the distribution function of $(x_1^{(N)}, \ldots, x_k^{(N)})$ and that $F_N \longrightarrow F$, as $N \longrightarrow \infty$, at all continuity points of F. If $g(x_1, \ldots, x_k)$ is a Borel measurable function such that the set $D(g)$, of discontinuity points of $g(x_1, \ldots, x_k)$, satisfies $P\{D(g)\} = 0$, under F, then the distribution function F_{Ng} of

$$g(x_1^{(N)}, \ldots, x_k^{(N)})$$

converges as $N \longrightarrow \infty$, to that of F_g where F_g is the distribution function of

$$g(x_1, \ldots, x_k) .$$

Lemma 4: (A proof is given by Rao, M. M. [34], p. 8.) Let a_n be a sequence of real numbers such that $\lim_{n\to\infty} a_n = \bar{a}$, $\bar{a}$ finite. Let $F_n(y)$ be a sequence of distribution functions such that $F_n \longrightarrow F$. Then if $\bar{a}$ is a continuity point of F, $\lim_{n\to\infty} F_n(a_n) = F(\bar{a})$.

Lemma 5: (Cramér [8], sec. 28.3) Let $x_1,\ldots,x_N$ be a sequence of independent observations from a population with distribution function $F(x)$.

Let $s^2 = \frac{1}{n} \sum_{i=1}^{N} (x_i - \bar{x})^2$, where $n = N-1$. Then $\sqrt{n}\,(s^2 - \sigma^2)$ is asymptotically normal $[0, \sigma^4(2 + \gamma_2)]$ as $N \longrightarrow \infty$, where γ_2 is given by the following function of the moments of distribution of x_i:

$$\gamma_2 = \beta_2 - 3 = \frac{\mu_4}{\mu_2^2} - 3 .$$

Proof: Cramér proves this lemma for the variate

$(s')^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2$. It follows from Lemma 1 that the variate

$s^2 = \frac{N}{N-1} (s')^2$ has the same limiting distribution, since $\frac{N}{N-1}$ converges to one as $N \longrightarrow \infty$.

Lemma 6: (Rao, C. R. [33], p. 208.) Let T_N be a sequence of real-valued statistics with the property that $\sqrt{N}[T_N - \theta]$ is asymptotically normally distributed $[0\ ,\ \psi(\theta)]$. Then if f is any function of T_N, it follows that $\sqrt{N}[f(T_N) - f(\theta)]$ is asymptotically normal $[0, (\frac{df}{d\theta})^2\, \psi(\theta)]$, provided that $\frac{df}{dT}$ is continuous in the neighborhood of θ.

<u>Lemma</u> 7: The limiting distribution of $\sqrt{n}(\ln s^2 - \ln \sigma^2)$ is the normal distribution with mean zero and variance $2 + \gamma_2$.

<u>Proof</u>: In Lemma 6, let $T_N = s^2$, $N = n$, $\Theta = \sigma^2$, $\varphi(\Theta) = \sigma^4(2 + \gamma_2)$, and $f(T_N) = \ln T_N$. It follows from Lemma 5 that T_N satisfies the hypothesis of Lemma 6 and the result follows.

The proof of the theorem below follows from results used by Andrews [2] in the proof of his Theorem 5.2.

<u>Theorem</u> 1: Let $H_1, H_2, \ldots$ denote a sequence of alternative hypotheses, where H_M denotes the hypothesis given by $\frac{\sum(\beta_k)^2}{M}$ and $\sum_{i=1}^{K} (\beta_k)^2$ is a constant. If the distribution function of z, say $F(z)$, satisfies the following conditions:

1) F possesses a continuous derivative F' except at most on a set S where $\int_S dF(z) = 0$,

2) There exists a function g which bounds the difference quotient $\left|[F(z + \Theta) - F(z)]/\Theta\right| \leq g(z)$ for which $\int g(z)dF(z) < \infty$,

3) The variance of z is finite,

then

$$F_L \longrightarrow \frac{1}{K-1} \chi'^2_{K-1}(\delta), \text{ in distribution, as } M \longrightarrow \infty ,$$

where $\delta^2 = \frac{\sum(\beta_k)^2}{\sigma_L^2}$, and σ_L^2 is the variance of $\ln s^2$.

In particular, under the hypothesis of variance homogeneity,

$$F_L \longrightarrow \frac{\chi^2_{(K-1)}}{(K-1)} \text{ in distribution} .$$

Corollary: Denote by $F_{\nu_1,\nu_2,\alpha}$ the upper 100 α percentage point of the F distribution with ν_1 and ν_2 degrees of freedom. Then

$$\lim_{M\to\infty} P\left\{F_L > F_{\nu_1,\nu_2,\alpha}\right\} = \alpha .$$

Proof: By definition we have

$$\int_{-\infty}^{F_{\nu_1,\nu_2,\alpha}} dF_{\nu_1,\nu_2} = 1 - \alpha . \qquad \ldots(1.22)$$

By the preceding theorem we have

$$F_L \longrightarrow F_{K-1\ ,\ \infty} = \chi^2_{K-1}/(K-1) . \qquad \ldots(1.23)$$

It can also be shown that

$$\lim_{\nu_2\to\infty} F_{\nu_1,\nu_2,\alpha} = F_{\nu_1,\infty,\alpha} . \qquad \ldots(1.24)$$

Since any subsequence of a convergent sequence also converges we have

$$\lim_{M\to\infty} F_{K-1,K(M-1),\alpha} = F_{K-1,\infty,\alpha} . \qquad \ldots(1.25)$$

We have shown that $\{F_\alpha\}$ is a sequence of numbers and that the distribution functions of F_L form a sequence of distribution functions which together satisfy the hypotheses of Lemma 4, and the result follows.

<u>Theorem 2</u>. If F(z) satisfies the conditions 1 - 3 of Theorem 1, then the distribution function of F_L converges to that of the non-central F-distribution as subgroup size tends to infinity. i.e.

$$F_L \longrightarrow F'_{K-1,K(M-1)}(\delta) \text{ in distribution as } A \longrightarrow \infty,$$

where $$\delta^2 = \frac{M\sum_{k=1}^{K}(\beta_k)^2}{2 + \gamma_2} \qquad \ldots(1.26)$$

In particular, under the hypothesis of variance homogeneity,

$$F_L \longrightarrow F_{K-1,K(M-1)} \text{ in distribution.}$$

<u>Proof</u>: Consider the variate

$$z'_{km} = \sqrt{A-1}\,[z_{km} - E(z_{km})]\ . \qquad \ldots(1.27)$$

We note that the distribution of F_L is invariant under this transformation. By Lemma 5, z'_{km} converges in distribution to that of a normal variate with mean zero and variance $(2 + \gamma_2)$. Since the z_{km} are independently distributed it follows that their joint distribution function converges to the joint distribution function of independent normal variates. The theorem then follows from an application of Lemma 3.

Corollary: $$P\left\{F_L > F_\alpha\right\} \longrightarrow \alpha \text{ as } A \longrightarrow \infty .$$

Proof: The corollary follows immediately from the definition of convergence in distribution since $F_{K-1,K(M-1),\alpha}$ is fixed.

Corollary: When the sampled population is non-normal, the asymptotic power of the L.V. test, as $A \longrightarrow \infty$, is $\lesseqgtr$ that which would be obtained under normal sampling, according as to whether $\gamma_2 \gtrless 0$.

Proof: It is easily shown that for any population $\gamma_2 \geq -2$. For a normal population $\gamma_2 = 0$. It can be shown that, for K and M fixed,

$$P\left\{F'_{K-1,\ K(M-1)}(\delta) > F_{K-1,K(M-1),\alpha}\right\}$$

is a monotone increasing function of δ,

where $$\delta^2 = \frac{M\sum_{k=1}^{K}(\beta_k)^2}{(2+\gamma_2)} .$$

If we also consider $\sum_{k=1}^{K}(\beta_k)^2$ fixed, then δ^2 is a monotone decreasing function in γ_2 and the result follows.

Theorem 3. Under the null hypothesis of variance homogeneity F_L converges to one in probability as K tends to infinity.

Proof: This theorem follows immediately from Lemma 2. It is well known that under the null hypothesis the numerator and the denominator of the variance ratio are unbiased estimates of the variance of the sampled variate. Since F_L is the ratio of two

quadratic forms it satisfies the hypotheses of Lemma 2 and the result follows.

Thus the asymptotic results which we have demonstrated are:

(i) Under H': $F_L \longrightarrow \frac{1}{K-1} \chi'^2_{K-1}(\delta)$ in distribution as $M \longrightarrow \infty$

$$\text{where } \delta^2 = \frac{\sum(\beta_k)^2}{\sigma_L^2}$$

Under H_0: $F_L \longrightarrow \frac{1}{K-1} \chi^2_{K-1}$ in distribution as $M \longrightarrow \infty$

$$\lim_{M \to \infty} P\{F_L > F_\alpha\} = \alpha$$

(ii) Under H': $F_L \longrightarrow F'_{K-1,K(M-1)}(\delta)$ in distribution as $A \longrightarrow \infty$

Under H_0: $F_L \longrightarrow F_{K-1,K(M-1)}$ in distribution as $A \longrightarrow \infty$

$$P\{F_L > F_\alpha\} \longrightarrow \alpha \text{ as } A \longrightarrow \infty$$

(iii) Under H_0: $F_L \longrightarrow 1$ in probability as $K \longrightarrow \infty$.

1.8. The Analysis of Variance of Logarithms of Ranges. A procedure is now proposed that may be used as an alternative to the L.V. test for testing the hypothesis of variance homogeneity. The method will be known as the log-range test, or L.R. test. In the following subsections the procedure will be described and justified, its use will be compared with that of the L.V. test, and some properties of the test will be discussed when the sampled distribution is rectangular as well as when it is normal.

1.8.1. The Log-range Test. The procedure for applying the L.R. test is directly analogous to that for the L.V. test (See section 1.1.):

(i) Divide the J observations y_{kj} within each of the K groups into M subgroups of size A. ($MA = J$, $M > 1$, $A > 1$)

(ii) Calculate the range of the observations in each subgroup, i.e., calculate

$$x_{km(A)} - x_{km(1)}, \qquad \ldots(1.28)$$

where $x_{km(A)}$ and $x_{km(1)}$ are respectively the largest and the smallest observation in the m-th subgroup of the k-th group.

(iii) Calculate the logarithm of the above range. Call this variate z^*_{km}. Thus

$$z^*_{km} = \log_{10} [x_{km(A)} - x_{km(1)}] \qquad \ldots(1.29)$$

(iv) Using z^*_{km} as a variate, carry out the usual one-way Analysis of Variance. Denote the test statistic by F^*_L.

(v) If $F^*_L > F_\alpha$ reject the null hypothesis given by

$$H_o : \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_K^2 .$$

In a previous technical note [26], section 2, the use of the logarithmic transformation applied to variances was justified. The same argument will be repeated here with regard to justification of the logarithmic transformation applied to range.

We have

$$z^*_{km} = \log_{10} R_{km} \qquad \ldots(1.30)$$

where $$R_{km} = [x_{km(A)} - x_{km(1)}] \ . \qquad \ldots(1.31)$$

Now it is mathematically convenient to consider the variate $2\ z^*_{km}$, and moreover also to consider the logarithmic transformation in terms of natural logarithms rather than those to the base 10. Clearly the theory of the test will be unaffected by such changes, and we may write

$$Z^*_{km} = \ln \frac{R^2_{km}}{\sigma_k^2} + \ln \sigma_k^{\ 2} \ . \qquad \ldots(1.32)$$

It follows that

$$E(Z^*_{km}) = E(\ln \frac{R^2_{km}}{\sigma_k^{\ 2}}) + \ln \sigma_k^{\ 2} \qquad \ldots(1.33)$$

$$= c + \ln \sigma_k^{\ 2} \ , \qquad \ldots(1.34)$$

where $$c = E(\ln \frac{R^2_{km}}{\sigma_k^{\ 2}}) \ .$$

Equations (1.32) and (1.34) show that testing the hypothesis of equality of means of the variate Z^*_{km} , or z^*_{km}, is equivalent to testing the hypothesis of variance homogeneity. It would be desirable for the transformed variable to have approximately a normal distribution. In section 1.7 it was demonstrated that this is true for the logarithm of variance transformation when A, the subgroup size, is large. However, it will be shown in 1.8.3 below, in particular, that when the sampled population is rectangular, the asymptotic distribution, as the sample size is increased without limit, of log-range is not normal.

1.8.2. <u>The L. R. Test for a Normal Population</u>. Some properties of the L.R. test when the sampled population is normal will now be discussed. It is well known that, except for samples of size 2, the distribution of range is expressible only in integral form. The limiting distribution of range is not known and it appears that the problem of finding the limiting distribution of the logarithm of the range is intractable. It is possible, however, to find the cumulants of the logarithm of the range by numerical methods.

The cumulative distribution of range has been computed and tabled by Harter and Clemm [17]. The density function of range was obtained by numerical differentiation of the cumulative distribution function. Finally, the first seven cumulants of log-range were obtained by numerical integration. The method was checked by computing the first four cumulants of range and comparing these with the tabled results of Harter and Clemm.

The first seven standardized cumulants for the distribution of $\ln R^2$ are given in Table 15. For comparison the corresponding cumulants of $\ln s^2$ are also given. The latter were computed from the tables of Davis [12]. For samples of size 2, $R^2 = 2s^2$ and in this case $E(\ln R^2) = \ln 2 + E[\ln s^2]$.

For $N = 2$ the cumulants of order two or larger are the same for the two variables. For N larger than 2 the standardized cumulants of $\ln R^2$ of order higher than two are closer to the normal theory value of zero than are the corresponding cumulants of $\ln s^2$, and the variance of $\ln R^2$ is larger than the variance of $\ln s^2$.

TABLE 15

A. Constants for the distribution of $\ln R^2$.

N	K_1	K_2	γ_1	γ_2	γ_3	γ_4	γ_5
2	-.5772	4.9348	-1.535	4.0000	-14.26	64.00	2176.00
4	1.2358	.9456	-.8659	1.3372	-2.49		
6	1.7392	.5184	-.6378	.8302	-1.60	4.00	
8	2.0056	.3652	-.4974	.5240	-.84	1.93	-5.17
12	2.3040	.2400	-.3404	.2623	-.33	.53	-1.58

B. Constants for the distribution of $\ln s^2$.

N	K_1	K_2	γ_1	γ_2	γ_3	γ_4	γ_5
2	-1.27036	4.9348	-1.535	4.000	-14.26	64.00	2176.00
4	-.36898	.9348	-.917	1.613	-4.11	13.60	262.20
6	-.21313	.4904	-.688	.931	-2.08	4.91	61.66
8	-.14961	.3309	-.570	.644	-1.08	2.41	22.35
12	-.09365	.1993	-.445	.395	-.52	.92	5.28

1.8.3. __The L.R. Test when the Sampled Distribution is Rectangular.__ In section 1.7, it was shown that as the subgroup size increases the distribution of the test statistic F_L converges to that of the normal theory F or F', according to whether the null or an alternative hypothesis is true. It will be demonstrated below, however, that, in general, this result is not true of the statistic F_L^* obtained in the L.R. test. In particular, it will be shown that when the sampled population has a rectangular distribution, the distribution of the standardized variable

$$\frac{z^* - E(z^*)}{\sigma_{z^*}}$$

converges to that of a linear function of a chi-square variable with four degrees of freedom.

For this case it will also be shown that var $[z^*]$ is of order $1/N^2$, where N is the sample, or subgroup, size. In section 1.7 it was seen that var [z] is of order 1/N. Also, it should be noted that whereas for the L.V. test $P\left\{F_L > F_\alpha\right\} \longrightarrow \alpha$ as subgroup size increases, in view of the convergence noted above, this does not hold true in general for the L.R. test.

The distribution of the logarithm of the range of a sample of size N drawn at random from a rectangular distribution will now be considered.

Let x be a random variable with probability density function

$$f(x) = 1 , \quad 0 < x \leq 1$$
$$= 0 , \quad \text{elsewhere} \qquad \ldots(1.35)$$

and let R denote the range of a sample of size N, $(N > 1)$, drawn randomly and independently from this population. Then the density function of R is given by

$$g(R) = N(N-1)R^{N-2}(1-R) \ , \ 0 < R \leq 1$$
$$= 0 \qquad , \ \text{elsewhere} \qquad \ldots(1.36)$$

If we let $z^* = \ln R$; then $R = e^{z^*}$, and $dR = e^{z^*}dz^*$, thus

$$g(z^*) = N(N-1)e^{(N-1)z^*}(1-e^{z^*}) \ , \ -\infty < z^* \leq 0$$
$$= 0 \qquad , \ \text{elsewhere} \qquad \ldots(1.37)$$

The characteristic function of z^* is then given by

$$\phi_{z^*}(t) = E(e^{itz^*}) = N(N-1)\int_{-\infty}^{0} e^{(it+N-1)z^*}(1-e^{z^*})dz^*$$

$$= N(N-1)\left[\int_0^\infty \exp\,[-(it+N-1)z^*]dz^* - \int_0^\infty \exp[-(it+N)z^*]dz^*\right]$$

$$= [(1 + \frac{it}{N})(1 + \frac{it}{N-1})]^{-1} \qquad \ldots(1.38)$$

and so the cumulant generating function of z^* is given by

$$\psi_{z^*}(t) = \ln \phi_{z^*}(t) = -\ln(1 + \frac{it}{N}) - \ln(1 + \frac{it}{N-1}) \qquad \ldots(1.39)$$

whence we find that the j-th cumulant of z^* is given by

$$K_j = (-1)^j\,(j-1)!\,[N^{-j} + (N-1)^{-j}]\ . \qquad \ldots(1.40)$$

In particular

$$K_1 = E(z^*) = -\frac{(2N-1)}{N(N-1)}$$

$$\text{and} \qquad K_2 = \text{var}\,(z^*) = \frac{1}{N^2} + \frac{1}{(N-1)^2} \qquad \ldots(1.41)$$

We note also, from (1.38) that

$$\lim_{N\to\infty} \phi_{z^*}(t) = 1\ . \qquad \ldots(1.42)$$

Therefore, it follows from the continuity theorem that z^* converges in probability to zero (Cramér [8], sec. 10.4).

Consider now the standardized variable

$$Z^* = \frac{z^* - E(z^*)}{\sigma_{z^*}}\ . \qquad \ldots(1.43)$$

Then $\phi_{Z^*}(t) = \exp\,[-it\,E(z^*)/\sigma_{z^*}]\,\phi_{z^*}[t/\sigma_{z^*}]$, and from (1.41) it is seen that

$$E(z^*)/\sigma_{z^*} = -(2N-1)(2N^2-2N+1)^{-1/2}$$

and therefore

$$\phi_{Z^*}(t) = \exp\,[it(2N-1)(2N^2 - 2N+1)^{-1/2}] \cdot$$
$$[1+it(N-1)(2N^2 - 2N+1)^{-1/2}]^{-1}[1 + itN(2N^2 - 2N+1)^{-1/2}]^{-1}. \qquad \ldots(1.44)$$

It follows that

$$\phi_{Z^*_0}(t) = \lim_{N\to\infty} \phi_{Z^*}(t) = \exp\,[\sqrt{2}\,it]\,[\,1 + \frac{it}{\sqrt{2}}]^{-2}\,. \qquad \ldots(1.45)$$

where Z^*_0 is the limiting value of Z^*, as $N \longrightarrow \infty$.

But the characteristic function of $u = (a\,\chi^2_\nu + b)$ is

$$\phi_u(t) = e^{itb}(1-2\,ita)^{-\nu/2}$$

whence it is seen that Z^*_0 is distributed as $[-\frac{1}{2\sqrt{2}}\chi^2_4 + \sqrt{2}]$.

Also it follows from the continuity theorem that Z^* converges in distribution to Z^*_0.

Thus it is seen that rather than tending to normality, std. log-range tends to a linear function of a chi-square variable with four degrees of freedom, when the sampled distribution is rectangular [0 , 1] .

However it is felt that further investigation of the properties of this test would be justified since its application is simpler than that of the L.V. test.

1.9. <u>An Alternative Approach to the Analysis of Variance of Variances</u>.

In the analysis of variance of logarithms of variances the observations are grouped and transformed as described in 1.2. The model thus becomes

$$z_{km} = \mu_k'' + \epsilon'_{km}$$

$$k = 1,\ldots,K$$

$$m = 1,\ldots,M \qquad \ldots(1.46)$$

where, under the assumption that the original ϵ_{km} are normal, the ϵ'_{km} are log χ^2_{A-1} variables. The hypothesis of variance homogeneity may then be written as

$$H_o : \mu_k'' = \mu'' , \; k = 1,\ldots,K. \quad (\mu'' \text{ unspecified})$$

This hypothesis can be split into a number of sub-hypotheses which together imply it. First a somewhat weaker hypothesis will be discussed. Then using this, the above more general hypothesis may be tested.

Let c_k be any given real numbers such that $\sum_{k=1}^{K} c_k = 0$, and define the contrast Θ, as

$$\Theta = \sum_{k=1}^{K} c_k \mu_k'' .$$

Then we wish to test the null hypothesis, H_o: $\Theta = 0$, against the alternative, H_1: $\Theta \neq 0$,

Note that if $\sigma_k = \sigma_{k'}$ for all k, k' so that $\mu_k'' = \mu_{k'}''$, then H_o is true for all c_k, but the converse is not necessarily true for a given set of c_k's. To test the above null hypothesis the following procedure may be applied. Transform (1.46), by averaging on m, into

$$\bar{z}_{k.} = \mu''_k + \bar{\varepsilon}'_{k.} \quad . \tag{1.47}$$

Clearly the $\bar{\varepsilon}_{k.}$, $k = 1,\ldots,K$, are independent identically distributed random variables. Let the estimated contrast be

$$\hat{\theta} = \sum_{k=1}^{K} c_k \, \bar{z}_{k.} \quad . \tag{1.48}$$

Since $E(\bar{\varepsilon}'_{k.})$ does not depend on μ''_k, it follows that $E(\hat{\theta}) = \theta$, where $\theta = 0$ if H_0 is true, and $\theta \neq 0$ otherwise. Consider

$$\hat{\theta} - \theta = \sum_{k=1}^{K} c_k \, \bar{\varepsilon}'_{k.} \quad . \tag{1.49}$$

Now the test consists of rejecting $H_0 : \theta = 0$ if

$$|\hat{\theta} - \theta| \quad \text{, or rather} \quad |\hat{\theta}|$$

is large, and accepting otherwise. Thus the problem is solved if the distribution of the random variable in (1.49) is obtained. This can easily be done; the technical details will not be included here since the problem seems capable of further extension.

This concludes our discussion on the Analysis of Variance of variances. It is clear that certain definite progress has been made towards solving the problem, but that there is still a need for further investigation into various aspects of the methods proposed.

2. A TRANSFORMATION PROCEDURE IN THE ANALYSIS OF VARIANCE

2.1. Transformations in the Analysis of Variance. The assumptions underlying the classical Analysis of Variance theory are that the cell populations be distributed normally, and that the variances within each group be equal. It has been suggested by several writers that the validity of the F-test may not be seriously affected by lack of normality per se, at least when the sample sizes are equal, but that the test is sensitive to variance heterogeneity. It is for this reason that the methods discussed in Part I of this report have been presented. However, in Part I it was assumed that the data could be described by one particular mathematical model. The assumptions made were that the variables within each subgroup were normally distributed, but that the within group variance was not necessarily the same for all the subgroups. Clearly in a general investigation of the problem of variance heterogeneity in the Analysis of Variance, it is necessary to study other models also.

A different hypothesis will now be considered, namely that the variables are no longer normally distributed, but that a transformation may be found that will transform their distribution to normality. It will be further assumed that variance heterogeneity is caused by there being a relationship between the mean and standard deviation within each group. Suitable transformations for stabilizing the variance, although not necessarily normalizing the variable, have been proposed for some particular cases by Curtiss, [9], and others more recently. However, it appears that no method of testing whether such a situation really exists has

been put forward. It is proposed in this section to suggest a procedure for testing whether to transform, and to discuss the possibility of deciding between different transformations.

The idea has been put forward that the transformation to be used might be one of a family of transformations, such as

$y = (x+c)^p$, where y is distributed normally (Tukey [39]).

This transformation may by definition be made to include the logarithmic transformation for the case $p = 0$. The problem, then, becomes one of estimating the parameters p and c from the data, carrying out the transformation, and analyzing the transformed data. However, the writers feel that since the sample sizes will generally be small, and hence the power to discriminate between different distributions low, it would be more advantageous to restrict the choice of transformation to between a few readily applied transformations - square root and logarithmic, for example. With samples of the sizes generally found in the Analysis of Variance, it is unlikely that much would be gained from considering many alternative transformations. In fact, a possible procedure would be to choose a transformation before testing whether or not to transform, and then having made the test, if it is decided to transform, the chosen transformation would be applied whatever the distribution of the data might be.

When there is a relationship between the mean and the variance of the within group distributions in the Analysis of Variance, it may be the result of the distributions being log-normal or Pearson Type III. (For a discussion of these frequency distributions

the reader is referred to [1] and [13].) In this case the samples tend to have many observations below the sample mean, and relatively close to it, while they often have fewer readings above the mean, many of them being a greater distance from it. This suggests that a suitable transformation should have the effect of condensing the upper tail of the distribution. Thus a suitable transformation might be the logarithmic transformation or the square-root transformation. It was decided that throughout this preliminary investigation the logarithmic transformation should be used. Then if the true distribution is log-normal and the parameters have been estimated from the data, it would be hoped that the transformed distribution would be approximately normal. Moreover, it may be seen from Table 15, page 43, of this report (or more fully in Table 1, page 130, of [3]) and that γ_1 and γ_2 for the distribution of log χ^2_ν are nearer to zero than the corresponding values for the distribution of χ^2_ν. (Note: $\gamma_1 = \gamma_2 = 0$ for a normal distribution.) Thus it seems that if the data is in fact distributed as a chi-square distribution, rather than as a log-normal distribution, the logarithmic transformation will still tend to normalize the data.

Thus it is proposed that the following procedure be adopted:

1. Test to determine whether a transformation should be made.
2. a) If it is decided not to transform, analyze the original data.

   b) If it is decided to transform, carry out the logarithmic transformation, and then analyze the transformed data.

The analysis having been completed on the transformed data, it will be necessary to interpret the conclusions in terms of the original data.

The possibility of obtaining a test based on the likelihood ratio criterion was investigated, but it was decided that the resulting procedure would be too cumbersome for most practical purposes. Instead, a simpler procedure for testing was sought, even if its power of detecting departure from the Analysis of Variance assumptions might be somewhat less than that of a test based on the likelihood ratio.

2.2 <u>To test the null hypothesis that the data be normally distributed</u>.

As has been stated in 2.1, the validity of the assumption of homogeneity of variance in the Analysis of Variance is frequently questioned because there appears to be a relationship between the mean and the variance of the population distribution. It is seen that this may result in actually testing for lack of normality within the samples, rather than for heterogeneity of variance between the samples.

Consider now the following three distributions of the random variable x, each of which has mean ξ, and variance σ_x^2.

i) Normal

$$p_N(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left\{-\frac{1}{2}\left(\frac{x-\xi}{\sigma_x}\right)^2\right\}$$

$$-\infty < x < +\infty \qquad \ldots(2.1)$$

The moments and cumulant-ratios of this distribution are:

$$\mu_1'(x) = \xi$$

$$\mu_2(x) = \sigma_x^2$$

$$\mu_3(x) = 0$$

$$\mu_4(x) = 3\ \sigma_x^4$$

$$\gamma_1(x) = 0$$

$$\gamma_2(x) = 0$$

ii) Log-normal

$$p_L(x) = \frac{1}{\sqrt{2\pi}\,\sigma_y}\ \frac{1}{x + c}\ \exp\left\{-\frac{1}{2}\left(\frac{\log[x+c] - \eta}{\sigma_y}\right)^2\right\}$$

$$-c < x < +\infty \qquad \ldots(2.2)$$

where $$\eta = \log\left\{\frac{\xi + c}{\sqrt{1 + \frac{\sigma_x}{(\xi+c)^2}}}\right\}$$

and $$\sigma_y^2 = \log\left\{1 + \frac{\sigma_x^2}{(\xi+c)^2}\right\}$$

For this distribution it may be shown that:

$$\mu_1'(x) = \xi = \exp\left[\eta + \tfrac{1}{2}\sigma_y^2\right] - c$$

$$\mu_2(x) = \sigma_x^2 = \exp\left[2\eta + \sigma_y^2\right]\left(\exp\left[\sigma_y^2\right] - 1\right)$$

$$\mu_3(x) = \exp\left[3\eta + \tfrac{3}{2}\sigma_y^2\right]\left(\exp\left[\sigma_y^2\right] - 1\right)^2\left(\exp\left[\sigma_y^2\right] + 2\right)$$

$$\mu_4(x) = \exp[4\eta + 2\sigma_y^2](\exp[\sigma_y^2] - 1)^2 (\exp[4\sigma_y^2] + 2\exp[3\sigma_y^2] + 3\exp[2\sigma_y^2] - 3)$$

$$\gamma_1(x) = (\exp[\sigma_y^2] - 1)^{1/2} (\exp[\sigma_y^2] + 2)$$

$$\gamma_2(x) = \exp[4\sigma_y^2] + 2\exp[3\sigma_y^2] + 3\exp[2\sigma_y^2] - 6$$

[It should be noted that if $\sigma_x = \lambda(\xi + c)$, where λ is constant for all ξ, σ_x, then σ_y is constant.]

iii) Root-normal. (The relationship between this distribution and a chi-square with one degree of freedom is readily seen.)

$$p_R(x) = \frac{1}{\sqrt{2\pi}\,\sigma_y} \frac{1}{\sqrt{x+c}} \exp\left\{-\frac{1}{2}\left(\frac{\sqrt{x+c} - \eta}{\sigma_y}\right)^2\right\}$$

$$-c < x < +\infty \qquad \ldots(2.3)$$

where $\eta^2 = [(\xi + c)^2 - \frac{1}{2}\sigma_x^2]^{1/2}$

$$\sigma_y^2 = (\xi + c) - [(\xi + c)^2 - \frac{1}{2}\sigma_x^2]^{1/2}$$

and for this distribution it is seen that:

$$\mu_1'(x) = \xi = \sigma_y^2 + \eta^2 - c$$

$$\mu_2(x) = \sigma_x^2 = 2\sigma_y^4 + 4\sigma_y^2\eta^2$$

$$\mu_3(x) = 8\sigma_y^6 + 24\sigma_y^4\eta^2$$

$$\mu_4(x) = 48\sigma_y^8 + 192\sigma_y^6\eta^2 + 3(2\sigma_y^4 + 4\sigma_y^2\eta^2)^2$$

$$\gamma_1(x) = \left(8 + 24\frac{\eta^2}{\sigma_y^2}\right)\left(2 + 4\frac{\eta^2}{\sigma_y^2}\right)^{-3/2}$$

$$\gamma_2(x) = (16 + 64\frac{\eta^2}{\sigma_y^2})\ (2 + 4\frac{\eta^2}{\sigma_y^2})^{-2}$$

It is seen from the above results that the values of γ_1 and γ_2 for the log-normal distribution are functions solely of σ_y^2, whereas for the root-normal distribution they are functions depending only upon η^2/σ_y^2. Thus it would appear that in order to test for normality of a given distribution with either of the above distributions as alternatives we may investigate either γ_1 or γ_2 of the population. Geary and Pearson [15] found that, except for very large samples, γ_2 was difficult to investigate; thus we shall now propose a method for testing whether $\gamma_1 = 0$. Now if a distribution has $\gamma_1 = 0$ then it is symmetrical about its mean, and therefore a criterion that tests for symmetry will at the same time test whether $\gamma_1 = 0$. Moreover it would seem justifiable to use a test for symmetry, since it is likely that the F-test in the Analysis of Variance would be more affected by skewness of the sampled distribution than by possible departure from normal kurtosis.

Let us assume then that observations have been drawn randomly and independently and are classified into k groups, there being n_t observations in the t-th group. It should be noted that the k groups may be arranged as a one-way classification or according to a more complicated model. Let the i-th observation in the t-th group be denoted by x_{ti}.

In addition let us denote the mean and variance of the distribution from which the t-th group was drawn by ξ_t and σ_t^2 respectively, and further let us assume that all the distributions are continuous,

have equal limits (which may be infinite) and have a third moment about their mean equal to the common value μ_3.

Then the null hypothesis that we really wish to test is that the x_{ti} are normally distributed, but it is decided that a weaker hypothesis will be tested, namely:

$$H_0 : \mu_3 = 0 \qquad \ldots(2.4)$$

against the alternative:

$$H' : \mu_3 > 0 . \qquad \ldots(2.5)$$

2.3. <u>Test procedure</u>. It is necessary in order to carry out the following procedure for there to be at least three replications in at least several of the groups, as will be seen below. The procedure is:

i) Calculate $\tilde{x}_{t.}$, the mean of the t-th group for $t = 1,\ldots,k$.

ii) Consider first those groups for which n_t is odd; to each group assign a characteristic random variable Y_t, such that

$Y_t = 1$ if the number of $(x_{ti} - \tilde{x}_{t.}) > 0$ is greater than $\frac{1}{2} n_t$

$Y_t = 0$ otherwise.

iii) In each of the groups for which n_t is even, choose at random one observation (for example this may be done by using a table of random numbers), then assign Y_t to each of these groups such that,

$Y_t = 1$ if the number of remaining $(x_{ti} - \tilde{x}_{t.}) > 0$ is greater than $\frac{1}{2}(n_t - 1)$

$Y_t = 0$ otherwise.

(Note: $\tilde{x}_{t.}$ may still be the mean of all n_t observations.)

iv) Calculate

$$Y = \sum_{t=1}^{k} Y_t$$

v) Rule: Reject H_0 if $Y \leq c_{k,\alpha}$

Since the Y_t are independent, and $P\{Y_t = 1 \mid H_0\} = \frac{1}{2}$, the values of $c_{k,\alpha}$ may be obtained from tables of the Binomial Probability distribution such as [35] or [37]. Suggested critical values are listed in Table 16, with the corresponding values of α, the probability of a Type I error.

It should be noted that since the x_{ti} all have continuous distributions, $P\{x_{ti} = \tilde{x}_{t.}\} = 0$. However owing to rounding off in practical examples it sometimes occurs that an individual observation is equal to its group mean, it is suggested that any such observation be disregarded when assigning the Y_t.

TABLE 16

CRITICAL VALUES FOR Y

k	$c_{k,\alpha}$	α
4	0	.0625
5	0	.0313
6	0	.0156
7	1	.0625
8	1	.0352
9	1	.0195
10	2	.0547
11	2	.0327
12	2	.0193

<u>Example</u>. As an example consider the data reproduced in Table 17, from Table 5 of [29]. This data represents a cross-classification

TABLE 17

DATA FOR TWO-WAY CROSS-CLASSIFICATION ANALYSIS OF VARIANCE.

(1,1)		(1,2)		(1,3)		(1,4)	
184.938		78.414		47.894		96.063	
19.708		897.009		116.280		118.629	
123.594		320.543		392.286		102.719	
205.404	164.90	1,493.690	438.27	126.849	251.96	242.745	184.70
39.291		535.935		275.618		298.563	
164.190	26,709.78	37.300	207,318.29	83.596	77,749.99	163.373	11,456.84
595.256		517.502		99.783		224.074	
55.757		157.912		95.583		111.386	
142.309		198.146		305.212		81.532	
118.510		146.204		976.529		407.884	
(2,1)		(2,2)		(2,3)		(2,4)	
43.816		104.167		220.527		976.529	
147.378		452.601		92.296		442.746	
33.549	67.51	851.506	239.16	625.784	457.66	124.462	527.30
22.265		151.263		189.049		703.449	
33.115	2,607.66	218.331	61,294.98	346.545	190,441.23	289.168	126,482.18
120.542		100.585		554.457		468.719	
36.598		35.198		1,576.271		1,226.605	
24.804		285.710		67.898		364.310	
64.651		75.566		376.524		127.485	
148.413		116.629		533.249		549.500	
(3,1)		(3,2)		(3,3)		(3,4)	
215.511		177.858		1,392.695		4,150.568	
39.252	66.92	159.336	191.39	1,047.322	1,622.58	1,094.443	3,923.31
27.167		112.730		538.621		8,901.720	
221.848	6,637.42	418.214	18,047.20	913.246	3,894,270.56	1,669.025	8,063,786.51
9.318		111.720		2,000.194		1,553.144	
16.216		214.858		1,626.193		5,937.240	
63.561		103.855		219.206		7,150.949	
30.386		112.168		220.750		1,085.717	
12.654		57.283		1,279.218		2,042.641	
33.315		445.863		6,988.347		5,647.686	

[The sample means and variances are at the right of each cell, the numbers (i,j) above each cell denote the cell number.]

model with three rows and four columns. The distribution of the population is known, and is in fact that of (2.2), the log-normal, with $\sigma_y = 1$, $c = 0$, and with the quantity η for any given cell made up of the sum of a row effect, a column effect, and an interaction.

In order to carry out the Y-test we notice that no observation is equal to its cell mean, and that there are ten observations in each cell. Thus it is necessary to ignore one observation from each cell when assigning the value of Y_t to the cell. (The choice was made at random.) The value of Y_t and the ignored observation for each cell are listed in Table 18 below.

TABLE 18

Y-test on data of Table 17

Cell	Y_t	Ignored observation
(1,1)	0	5
(1,2)	0	5
(1,3)	0	7
(1,4)	0	3
(2,1)	0	10
(2,2)	0	5
(2,3)	0	5
(2,4)	0	9
(3,1)	0	1
(3,2)	0	1
(3,3)	0	3
(3,4)	1	9

Thus we obtain $Y = \sum_{t=1}^{12} Y_t = 1$, which from Table 16 is seen to be significant, with $\alpha = .02$ approximately.

The null hypothesis of symmetry is therefore rejected, and it is decided to apply the logarithmic transformation to the data

before carrying out the Analysis of Variance.

2.4. <u>Further work</u>. Having decided to apply to the data a transformation of the form

$$y = \log (x + c) \qquad \ldots(2.6)$$

we must either know c a priori, or must estimate it from the data. ($-c$ is the lower limit of the distributions.)

Work being carried out at present includes investigation of methods of estimating c.

It is proposed to investigate the power of the Y-test against particular forms of alternative distribution, for various sample sizes, by a Monte Carlo procedure. It is hoped also to investigate the possibility of being able to make a test on the data which would show whether the logarithmic or the square-root transformation is the more appropriate for the particular data under consideration. However, a preliminary study has suggested that residual variation may frequently obscure which form of transformation should be the more appropriate.

Finally, it should be noted that the Y-test procedure requires there to be a reasonable number of cells having at least three observations in them. When these conditions are not satisfied some alternative procedure is required; one possibility that should be investigated is that of always transforming to standard normal scores when the observations are such that no test of normality may be made.

3. MISCELLANEOUS TOPICS

The topics of the following sections have been investigated, and although it was felt that each one justified inclusion in this report, it was decided that none warranted an individual main section.

3.1. <u>Correlated variables and transformations</u>.* In applying transformations, it has almost always been assumed that the observations, in the sample under consideration, are independently distributed but that their common distribution is not normal. Consequently, a transformation which makes them "nearly normal" is applied before the Analysis of Variance of the data is performed. It is not uncommon, however, that the observations are correlated in some manner, and if so the transformations for this type of variable lead to decidedly more difficult problems. In these circumstances, it may be of interest to transform the data so as to obtain "nearly normal" correlated data and carry on the analysis. The distribution problems for the correlated normal variables are not as completely solved as in the independent case. (Cf. for example, [7], [16] and [19].) The solutions to these problems are needed, however, as complementary to, and before application of, the transformations. Some of these questions have been considered recently, in a different manner, by Mizel and Rao [25], where the solution to the problem on correlated

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* The authors are grateful to V. J. Mizel and M. M. Rao, Carnegie Institute of Technology, Pittsburgh, Pa., who provided this section. Their conclusions were included in a joint paper [25], read by M. M. Rao before the annual meeting of the American Mathematical Society, January 1961, in Washington, D. C.

normal variables is deduced from a somewhat more general result. That solution in the present context will be briefly described.

Let $x = (x_1, \ldots, x_n)$ be a correlated sample from a normal distribution whose mean vector is μ and covariance matrix is Σ (both of n-th order). Suppose the breakdown of the sum of squares is done analogous to the one in the case of independence. Formally, this means that a quadratic form $Q = xAx'$ is broken up into several other forms, for instance, Q_1 and Q_2. Let $Q_i = xA_ix'$, so that, since

$$Q = Q_1 + Q_2 \qquad \ldots(3.1)$$

we have $A = A_1 + A_2$. If Q is known to be distributed as a chi-square (central or not makes no difference in what follows), with, say, n_o degrees of freedom, then the problem is this: for what types of A_1 and A_2 are Q_1 and Q_2 independently distributed as chi-square variables.

It may be seen that this problem is related to the results about and extending Cochran's theorem. For a detailed discussion and some extensions of the classical problem reference may be made to [7] and [16]. In a recent note [19], it was shown that if A_2 is a non-negative matrix, in (3.1) above, and if Q and Q_1 are distributed in chi-square form, then Q_2 is also a chi-square variable. At this point, it is natural to inquire how far this condition may be relaxed, or alternate (and possibly weaker) conditions may be obtained. (It is known that a quadratic form in normal variables, such as Q, is distributed as a (non-central) chi-square variable if and only if

$(A\Sigma)^2 = A\Sigma$, i.e. $A\Sigma$ is idempotent.) In [4], this problem is characterized as follows. If, in (3.1), Q and Q_1 are each distributed as chi-square variables with n_o and n_1 $(< n_o)$ degrees of freedom, then in order that Q_2 be distributed as a chi-square variable with $n_2 (= n_o - n_1)$ degrees of freedom independently of Q_1, it is necessary and sufficient that A_2 and Σ be commutative and that $A_2\Sigma$ be positive semi-definite (i.e. $A_2\Sigma = \Sigma A_2 \geq 0$). This implies that, when the covariance matrix is given (it should be known in all the problems of this type or the experimenter should have some idea about it), the breakdown such as (3.1) has to be done in a somewhat more careful manner than in the independence case. The significance of this is then that the transformation problem for the correlated observations is more difficult, in general, and consequently more care should be given in its treatment when the transformation of data is contemplated at all.

3.2. Tests on the mean of a non-normal variable.* Suppose that it is desired to test the hypothesis $H_o : \mu_x = {}_o\mu_x$ against the alternative hypothesis $H_1 : \mu_x = {}_1\mu_x$, where x is a continuous random variable with mean μ_x and variance σ_x^2, whose distribution function is not normal.

Suppose also that there exists a transformation $y = (x + c)^p$ which normalizes x for some constants c and p, (y is distributed normally with mean μ_y and variance σ_y^2).

The objects of this investigation are (i) to determine whether x has an asymptotically normal distribution (as the coefficient of variation approaches zero), so that the usual normal

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* This investigation is being conducted by Miss C. D. Trammell.

theory tests may be applied when the mean μ_x is large relative to the standard deviation σ_x, and (ii) to find a method for estimating "best" (in some sense of the word) values of c and p for a given set of data.

For $p = 2k + 1$, k being a positive integer, the transformation $y = (x + c)^p$ is continuous and monotone, since

$$\frac{dy}{dx} = (2k + 1)(x + c)^{2k} > 0 , \quad x > -c \qquad \ldots(3.2)$$

Thus a theorem of Olds and Severo [30], p. 36, is applicable to this case. This theorem gives the most powerful critical region of size α for testing the hypothesis $H_0 : \mu_x = {}_0\mu_x$ against the alternative $H_1 : \mu_x = {}_1\mu_x > {}_0\mu_x$.

The case for $p = \frac{1}{2}$ was considered by M. M. Rao, [34], with σ_y^2 fixed.

It is hoped that results will be obtained for the case where $p = \frac{1}{n}$, n being a positive integer.

3.3 Some results on the moments of sample range. In Part I of this report two test procedures were proposed, namely, the log-variance test and the log-range test. In developing the theory for these tests it was assumed that within each group the observations were distributed normally and independently. Each group was subdivided and, according to which test was being used, the logarithm was calculated either of the variance or of the range of each subgroup. An Analysis of Variance was then performed on these quantities. An investigation was initiated to consider the effect upon these tests

if the observations within each subgroup, whilst being normally distributed, were not independent. Thus it was assumed that the joint distribution of the variables within a subgroup was multivariate normal, with not all the correlations equal to zero. No useful progress has been made on this topic as such; however, some general results have been obtained for the mean and variance of sample range. It was decided to include these results in this report since they are of interest in their own right.

The results show that if range is used to estimate the population standard deviation, assuming the variables to be independent when they are in fact correlated, then the estimate will be seriously biased even if the correlation between the sample variables is small. The bias is increased if the result is used to estimate the standard deviation of the distribution of sample means, still assuming independence.

It is clear that these results might be used to modify the standard Statistical Quality Control techniques when it cannot be assumed that the observations within samples are independent.

The method used for investigation of the distribution of sample range when the variables are independent (see [17, Vol. 1, p. vi] or [32, p. 43]) cannot be extended to the situation where the variables are correlated. This is clear since, given the value of one variate, the values of all the others are no longer independent of it. Thus a different technique must be used. The following discussion will be confined to obtaining the moments of sample range in the case where the variates may be correlated.

3.3.1. A general expression for the moments of sample range. A sample of size n is drawn from a continuous distribution such that the joint density function of the sample variates $x_1,\ldots,x_n$, where for instance the subscripts denote the order of drawing, is

$$p(x_1,\ldots,x_n) .$$

There are then n! possible ways in which the subscripts $1,\ldots,n$ may occur when the sample is ordered with respect to the magnitude of the variates. One such ordering may be written

$$x_i > x_j > \ldots > x_p > x_q > x_t . \qquad \ldots(3.3)$$

Then if we denote the sample range by R, the s-th moment of R about zero will be given by the sum of n! integrals (one for each possible ordering) of the form:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{x_i}\ldots\int_{-\infty}^{x_p}\int_{-\infty}^{x_q}(x_i - x_t)^s \, p(x_1,\ldots,x_n)dx_t dx_q\ldots dx_j dx_i \qquad \ldots(3.4)$$

which may be written as

$$E[(x_i - x_t)^s \mid x_i > x_j > \ldots > x_p > x_q > x_t] \, P\{x_i > x_j > \ldots > x_p > x_q > x_t\}.$$

For example, let us denote the variates of a sample of size

three by x, y, z, whose joint density function is

$$p(x,y,z)$$

and let us further denote the sample range by $R(x,y,z)$; then

$$\begin{aligned}
E\left\{[R(x,y,z)]^s\right\} = {} & E[(x-y)^s \mid x > z > y] \quad P\left\{x > z > y\right\} \\
& + E[(x-z)^s \mid x > y > z] \quad P\left\{x > y > z\right\} \\
& + E[(y-z)^s \mid y > x > z] \quad P\left\{y > x > z\right\} \\
& + E[(y-x)^s \mid y > z > x] \quad P\left\{y > z > x\right\} \\
& + E[(z-x)^s \mid z > y > x] \quad P\left\{z > y > x\right\} \\
& + E[(z-y)^s \mid z > x > y] \quad P\left\{z > x > y\right\}. \qquad \ldots(3.5)
\end{aligned}$$

From this expression the moments of range may be calculated if the necessary integrals may be evaluated exactly, or else approximations to the moments may be obtained by evaluating the integrals by numerical methods.

An interesting result is obtained for the mean range in a sample of size three by putting $s = 1$ in (3.5). If we then write

$$E(x-y) = \tfrac{1}{2}[E(x-y) + E(x-z+z-y)] = \tfrac{1}{2}[E(x-y) + E(x-z) + E(z-y)] \,, \qquad \ldots(3.6)$$

it may readily be shown that

$$E[R(x,y,z)] = \tfrac{1}{2}\left\{E[R(x,y)] + E[R(y,z)] + E[R(z,x)]\right\} \qquad \ldots(3.7)$$

where $R(x,y)$ denotes the range in a sample of size two, whose variates x,y have a joint density function

$$p(x,y) = \int_{-\infty}^{+\infty} p(x,y,z)\, dz \qquad \ldots(3.8)$$

Thus if the mean value of range in a sample of size two is obtainable then it is always possible to deduce the mean range for a sample of size three. To obtain the former is frequently a trivial matter. Note: This result does not depend upon the form of $p(x_1,\ldots,x_n)$.

3.3.2. <u>Moments of sample range when the distribution is multivariate normal.</u> In the following it will be assumed that the sample variates $x_1,\ldots x_n$ have jointly a multivariate normal distribution with

$$E(x_i) = 0$$

$$\text{var } (x_i) = \sigma_i^2$$

$$E(x_i x_j) = \sigma_i \sigma_j \rho_{ij} \quad \text{for } i \neq j \, . \qquad \ldots(3.9)$$

(i) Sample size : $n = 2$. It may easily be shown that

$$E(R) = \sqrt{\frac{2}{\pi}}\,[\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2]^{1/2} \qquad \ldots(3.10)$$

$$\text{var } (R) = (1 - \frac{2}{\pi})\,[\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2] \quad \ldots(3.11)$$

and if $\sigma_1 = \sigma_2 = \sigma$ these reduce to

$$E(R) = \frac{2\sigma\sqrt{1-\rho_{12}}}{\sqrt{\pi}} \quad \text{and var } R = 2(1 - \frac{2}{\pi})(1 - \rho_{12})\sigma^2 \quad \ldots(3.12)$$

Now if R is used to estimate σ, assuming $\rho_{12} = 0$, we use (see [32, p. 46])

$$\hat{\sigma} = \frac{R}{d_2} \qquad \ldots(3.13)$$

Thus if there is in fact correlation

$$E(\hat{\sigma}) = \sigma\sqrt{1 - \rho_{12}} \qquad \ldots(3.14)$$

Moreover, if, as in Quality Control techniques, sample range is used to estimate the standard deviation of the sample mean, assuming independence, the following would be obtained:

$$E(\hat{\sigma}_{\bar{x}}) = \sigma_{\bar{x}}\sqrt{\frac{1 - \rho_{12}}{1 + \rho_{12}}} \, . \qquad \ldots(3.15)$$

Also if the variance of the range is estimated on the assumption of independence, the following relationship holds:

$$\text{var}\,(R|\rho) = (1 - \rho)\,(\text{var}\,[R|\rho = 0]) \, . \qquad \ldots(3.16)$$

(ii) Sample size: n = 3

It is immediately obtained from (3.7) and (3.10) that

$$E(R) = \frac{1}{\sqrt{2\pi}}\Big\{[\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2]^{1/2} + [\sigma_2^2 - 2\rho_{23}\sigma_2\sigma_3 + \sigma_3^2]^{1/2}$$

$$+ [\sigma_3^2 - 2\rho_{31}\sigma_3\sigma_1 + \sigma_1^2]^{1/2}\Big\} \qquad \ldots(3.17)$$

It is possible to obtain expressions in closed form for

(a) var $(R|n = 3)$

and (b) E $(R|n = 4)$

however, very heavy algebra is involved.

If it is assumed that $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$, and (3.17) is

used to estimate $\sigma_{\bar{x}}$, it is seen that

$$\lambda = \frac{E(\hat{\sigma}_{\bar{x}})}{\sigma_{\bar{x}}} = \frac{[\sqrt{1-\rho_{12}} + \sqrt{1-\rho_{23}} + \sqrt{1-\rho_{31}}]}{\sqrt{3}\ [3 + 2\rho_{12} + 2\rho_{23} + 2\rho_{31}]^{1/2}} \qquad \ldots(3.18)$$

Some values of this ratio are given in Table 19.

(iii) The case for general values of n, when all the correlations are equal, has been considered by Hartley, [18]. In addition the probability integral for max x_i has been tabulated by Kudô, [22], for the case when the correlations are all equal.

Hartley demonstrated that if the means, μ, and variances, σ^2, of all variates are equal, and if

$$\rho_{ij} = \rho > -1/n-1 \ , \ i \neq j \ ,$$

then the sample range is exactly distributed as the range in a sample of n independent normal variates with variance $\sigma^2(1-\rho)$, and further, is distributed independently of mean $\bar{\bar{x}}$.

Thus it is seen that

$$E(R|\rho) = \sqrt{1-\rho} \quad E(R|\rho = 0)$$

and

$$\text{var}\ (R|\rho) = (1-\rho)\ \text{var}\ (R|\rho = 0)\ . \qquad \ldots(3.19)$$

Thus it may be shown that

$$\lambda = \frac{E(\hat{\sigma}_{\bar{x}})}{\sigma_{\bar{x}}} = \frac{\sqrt{1-\rho}}{\sqrt{1+(n-1)\rho}} \text{ , for } \rho > -1/n-1 \qquad ..(3.20)$$

The effect of this for n = 5 is noted as follows:

$\rho = .05 \qquad \lambda = .89$

$\rho = .20 \qquad \lambda = .67$

$\rho = .375 \qquad \lambda = .5$

TABLE 19

VALUES OF λ.

ρ	n = 2	n = 3: $\rho_{12}=\rho_{23}=\rho_{31}=\rho$	n = 3: $\rho_{12}=\rho_{23}=\rho;\rho_{13}=\rho^2$
-1.0	∞		1.633
-.5	1.732	∞	2.041
0	1.000	1.000	1.000
+.2	.816	.756	.811
.4	.655	.577	.642
.5	.577	.500	.561
.6	.500	.426	.486
.8	.333	.277	.315
.9	.229	.189	.215
+1.0	0	0	0

Clearly there is a wide field here both for further investigation and for the application of these results. Until methods of using these results are proposed, the results should act as a warning to those who use the sample range to estimate population parameters without considering whether or not the sample variates are independent.

BIBLIOGRAPHY

[1] Aitchison, J. and Brown, J. A. C. The Lognormal Distribution, Cambridge University Press, 1957

[2] Andrews, F. C. Asymptotic behavior of some rank tests for analysis of variance. Ann. Math. Stat. 25, (1954), pp. 724-735.

[3] Bartlett, M. S. and Kendall, D. G. The statistical analysis of variance-heterogeneity and the logarithmic transformation. J.R.S.S. (Supp.) 8, (1946), pp. 128-138.

[4] Barton, D. E. and Dennis, K. E. The conditions under which the Gram-Charlier and Edgeworth curves are positive definite and unimodal. Biometrika 39, (1952), pp. 425-427.

[5] Box, G. E. P. Non-normality and tests on variances. Biometrika 40, (1953), pp. 318-335.

[6] Box, G. E. P. and Andersen, S. L. Permutation theory in the derivation of robust criteria and the study of departures from assumption. J.R.S.S., B, 17 (1955), pp. 1-26.

[7] Chipman, J. S. and Rao, M. M. On the Use of Idempotent Matrices in the Treatment of Linear Restrictions in Regression Analysis. Technical Report No. 10, Department of Statistics, University of Minnesota, 1959.

[8] Cramer, H. Mathematical Methods of Statistics. Princeton University Press, 1952.

[9] Curtiss, J. H. On transformations used in the analysis of variance. Ann. Math. Stat. 14, (1943), pp. 107-122.

[10] David, F. N. The z-test and symmetrically distributed random variables. Biometrika 46, (1959), pp. 123-129.

[11] David, F. N. and Johnson, N. L. The effect of non-normality on the power function of the F-test in the analysis of variance. Biometrika 38, (1951), pp. 43-57.

[12] Davis, H. T. Tables of Higher Mathematical Functions, Volumes 1-2. Principia Press, Inc. 1935.

[13] Elderton, W. P. Frequency Curves and Correlation. Cambridge University Press. (Recently issued by Dover Publications, Inc.)

[14] Gayen, A. K. The distribution of the variance ratio in random samples of any size drawn from non-normal universes. Biometrika 37, (1950), pp. 236-255.

[15] Geary, R. C. and Pearson, E. S. Tests of Normality. Biometrika Office, University College, London, 1938.

[16] Graybill, F. A. and Marsaglia, G. Idempotent matrices and quadratic forms in the general linear hypothesis. Ann. Math. Stat. 28, (1957), pp. 678-686.

[17] Harter, H. L., Clemm, D. S. The Probability Integrals of the Range and of the Studentized Range.--Probability Integral, Percentage Points, and Moments of the Range. Technical Report 58 - 484, Vol. 1, Wright Air Development Center, 1959.

[18] Hartley, H. O. The use of range in analysis of variance. Biometrika 37, (1950), pp. 271-280.

[19] Hogg, R. V. and Craig, A. T. On the decomposition of certain chi-square variables. Ann. Math. Stat. 29, (1958), pp. 608-610.

[20] Kaskey, G; Kolman, B.; Krishnaiah, P. R; and Steinberg, L. Statistical Techniques in Transistor Evaluation. Technical Report 1: Transformations to Normality. Remington Rand Univac , Philadelphia; Department of the Navy, Bureau of Ships, Contract NObs-72660, Jan. 1961.

[21] Kendall, M. G. and Stuart, A. The Advanced Theory of Statistics, Vol. 1, Hafner, 1958.

[22] Kudô, A. On the distribution of the maximum value of an equally correlated sample from a normal population. Sankhya 20, (1958), pp. 309-316.

[23] Levene, H. Robust tests for equality of variances. Contributions to Probability and Statistics, Stanford University Press, 1960, pp. 278-292.

[24] Mann, H. B. and Wald, A. On stochastic limit and order relationships. Ann. Math. Stat. 14, (1943), pp. 217-226.

[25] Mizel, V. J. and Rao, M. M. Nonsymmetric Projections in Hilbert Space. Technical Report No. 47, Department of Mathematics, Carnegie Institute of Technology, 1960.

[26] Odeh, R E. and Olds, E.G. Notes on the Analysis of Variance of Logarithms of Variances. WADC Technical Note 59-82, Wright Air Development Center, March 1959.

[27] Olds, E. G. and Kennard, R. W. On the Determination of Significance Levels of Control Charts for Continuous Variates. Technical Report No. 2, Contract No. DA-36-061-ORD-447, Department of Mathematics, Carnegie Institute of Technology, 1955.

[28] Olds, E. G. and Lewis, J. S. Notes on the Use of F in the Analysis of Variance of Attributes Data and on the Transformation of Measurements Data to Attributes Data. ARL Technical Note 60-118, Aeronautical Research Laboratories, 1960.

[29] Olds, E. G.; Mattson, T. B.; and Odeh, R. E. Notes on the Use of Transformations in the Analysis of Variance. WADC Technical Report 56-308, Wright Air Development Center, 1956.

[30] Olds, E. G. and Severo, N. C. A Comparison of Tests on the Mean of a Logarithmico Normal Distribution with Known Variance. WADC Technical Note 55-249, Wright Air Development Center, 1955.

[31] Pearson, E. S. and Hartley, H. O. Charts of the power function of the analysis of variance tests, derived from the non-central F-distribution. Biometrika 38, (1951), pp. 112-130.

[32] Pearson, E. S. and Hartley, H. O. (editors.) Biometrika Tables for Statisticians, Volume I. Cambridge University Press, 1954.

[33] Rao, C. R. Advanced Statistical Methods in Biometric Research. Wiley, 1952.

[34] Rao, M. M. Some Asymptotic Results on Transformations in the Analysis of Variance. ARL Technical Note 60-126, Aeronautical Research Laboratories, 1960.

[35] Romig, H. G. 50-100 Binomial Tables, Wiley, 1953.

[36] Scheffé, H. The Analysis of Variance. Wiley, 1959.

[37] Statistical Engineering Laboratory, Tables of the Binomial Probability Distribution. National Bureau of Standards, App. Math. Series 6.

[38] Tang, P. C. The power function of the analysis of variance tests with tables and illustrations of their use. Statistical Research Memoirs 2, (1938), pp. 126-157.

[39] Tukey, J. W. On the comparative anatomy of transformations. Ann. Math. Stat. 28, (1957), pp. 602-632.

Carnegie Institute of Technology, Pittsburgh, Pennsylvania, ON SOME TRANSFORMATIONS IN THE ANALYSIS OF VARIANCE, by E. G. Olds, M.Branson, R. E. Odeh. December 1961. 75 p. incl. illus. and tables. Contract AF 33(616)-3878 (Project 7071, Task 70429). ARL 89.
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